PMM U.S.S.R.,Vol.46,pp.129-131
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0021-8928/83/1 0129 $7.50/0

UDC 532.516:536.25

STABILITY OF THERMOCAPILLARY CONVECTION IN A FLUID FILLING A HALF-SPACE*

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An exact axisymmetric solution of the nonlinear problem of steady state thermocapillary convection caused by the action of a constant intensity point heat source situated at the free surface of a fluid filling a half-space, was obtained in the work /1/. Below the stability of this motion with respect to azimutically periodic perturbations is studied.

1. Let a point heat source of constant intensity Q be situated at the surface $z=0$ of a fluid filling the half-space $z>0$. The complete system of equations and boundary conditions describing the capillary convection appearing in the absence of a gravitational field is given in /1/. In the spherical coordinate system (whose origin coicides with the heat source and the polar axis with the z-axis), the system has the form

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\mathbf{v}=-\frac{1}{\rho}\nabla p+\nu\Delta\mathbf{v}, \quad \frac{\partial T}{\partial t}+\mathbf{v}\nabla T=\chi\Delta T, \quad \nabla\mathbf{v}=0 \tag{1.1}$$

$$\vartheta=\pi/2, \quad r\neq 0, \quad v_\vartheta=\frac{\partial T}{\partial\vartheta}=0, \quad \frac{\eta}{r}\frac{\partial v_r}{\partial\vartheta}=\frac{\partial\sigma}{\partial T}\frac{\partial T}{\partial r}, \quad \frac{\eta}{r}\frac{\partial v_\varphi}{\partial\vartheta}=\frac{\partial\sigma}{\partial T}\frac{1}{r}\frac{\partial T}{\partial\varphi} \tag{1.2}$$

$$\vartheta=0, \quad v_\vartheta=v_\varphi=0 \tag{1.3}$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi/2}\left(v_r T-\chi\frac{\partial T}{\partial r}\right)r^2\sin\vartheta\, d\vartheta=\frac{\chi}{\varkappa}Q \tag{1.4}$$

Here (1.2) represent the conditions at the free surface, (1.3) are the symmetry conditions at the polar axis, (1.4) is the integral condition of the constancy of the heat flux through a half-sphere with the center at the coordinate origin, η and ν denote the dynamic and kinematic viscosity, χ and $\varkappa$ are the thermal diffusivity and conductivity coefficients, ρ is the fluid density and σ is the surface tension coefficient which we shall assume to be linearly dependent on temperature. At infinity the velocity $\mathbf{v}(v_r, v_\vartheta, v_\varphi)$ and temperature T should both become zero.

The axially symmetric steady state solution v_0, T_0 of the problem (1.1)—(1.4) has the form /1/

$$v_{0r}=\frac{\Theta_1(\vartheta)}{r}, \quad v_{0\vartheta}=\frac{\Theta_2(\vartheta)}{r}, \quad v_{0\varphi}=0, \quad T_0=\frac{\Theta_3(\vartheta)}{r}$$

and the expressions for Θ_i $(i=1,2,3)$ are given in /1/.

2. Let us investigate the stability of this motion with respect to monotonous perturbations. Following the standard methods we superimpose on the basic motion $\mathbf{v}_0$, T_0 p_0 small perturbations of velocity $\mathbf{v}(r,\vartheta,\varphi)\exp(-\lambda t)$, temperature $T_1(r,\vartheta,\varphi)\exp(-\lambda t)$ and pressure $q(r,\vartheta,\varphi)\exp(-\lambda t)$. The critical values of the heat source intensity at which the basic flow collapses, correspond to the zero values of the decrements $\lambda=0$. In this case the system (1.1) linearized with respect to the perturbations, assumes the form

$$\mathbf{f}\equiv(\mathbf{v}_0\nabla)\mathbf{v}+(\mathbf{v}\nabla)\mathbf{v}_0-\nu\Delta\mathbf{v}=-\frac{1}{\rho}\nabla q \tag{2.1}$$
$$\mathbf{v}\nabla T_0+\mathbf{v}_0\nabla T_1=\chi\Delta T_1, \quad \nabla\mathbf{v}=0$$

with the boundary conditions (1.2), (1.3) for the perturbations remaining unchanged. The linearized integral condition for the perturbations (1.4), is equal zero.

We shall seek the solution of the problem (2.1), (1.2)—(1.4) with help of the Galerkin method, using as the approximation function

$$\mathbf{v}=a_1\mathbf{v}_1+a_2\mathbf{v}_2 \tag{2.2}$$

*Prikl.Matem.Mekhan.,46,No.1,pp.162-165,1982 the following expressions $(n = 1, 2)$:

$$\begin{aligned} v_{nr} &= r^{-1} u_n \cos m\varphi, \quad u_n = (1 - \cos\vartheta)^n [1 - (n+2)\cos\vartheta] \\ v_{n\vartheta} &= r^{-1} v_n \cos m\varphi, \quad v_n = \operatorname{ctg}\vartheta\, (1 - \cos\vartheta)^{n+1} (1 + m^2) \\ v_{n\varphi} &= r^{-1} w_n \sin m\varphi, \quad w_n = m (1 - \cos\vartheta)^n [1 - (n+2)\cos\vartheta] \sin\vartheta \\ T_1 &= r^{-1} g_1(\vartheta) \cos m\varphi \end{aligned} \tag{2.3}$$

The functions (2.3) satisfy all boundary and integral conditions, and the equation of continuity. Substituting (2.3) into (2.1) and carrying out the necessary manipulations (a prime denotes differentiation with respect to ϑ) , we obtain

$$f_r = \sum_{n=1}^{2} a_n \cos m\varphi \left\{ v_n \Theta_1' + u_n' \Theta_2 - 2u_n \Theta_1 - 2v_n \Theta_2 - \nu \left(u_n'' - \frac{m^2 u_n}{\sin^2\vartheta} + \operatorname{ctg}\vartheta \cdot u_n' \right) \right\} r^{-3} \tag{2.4}$$

$$f_\vartheta = \sum_{n=1}^{2} a_n \cos m\varphi \left\{ (\Theta_2 v_n)' - \frac{\nu}{1+m^2} \left[(m^2 - 1) v_n' + (3m^2 - 1) v_n \operatorname{ctg}\vartheta + \frac{(1 - m^4) v_n}{\sin^2\vartheta} - 2m^2 v_n \right] \right\} r^{-3}$$

$$f_\varphi = \sum_{n=1}^{2} a_n \sin m\varphi \left\{ \Theta_2 (w_n \sin\vartheta)' - \nu \left[(w_n' \sin\vartheta)' + (m^2 - 1) \frac{w_n}{\sin\vartheta} + 2m v_n' \right] \right\} \sin^{-1}\vartheta \cdot r^{-3}$$

We eliminate the pressure q from (2.1) by taking curl of both sides of the Navier—Stokes equation

$$\Delta \times \mathbf{f} = 0 \tag{2.5}$$

Substitution of (2.2) into (2.5) yields a discrepancy. We obtain the coefficients a_n $(n = 1, 2)$ of expansion in (2.2) from the condition that the discrepancy is orthogonal to the basis functions v_n

$$\int_0^{\pi/2} (\nabla \times \mathbf{f})\, \mathbf{v}_n \sin\vartheta\, d\vartheta = 0 \quad (n = 1, 2) \tag{2.6}$$

Since the expressions for Θ_1 and Θ_2 are bulky, approximate expressions are more convenient for numerical computations, and we choose the following expressions:

$$\Theta_1 = {}^1\!/_2 A\nu\, (1 - 2\cos\vartheta), \quad \Theta_2 = {}^1\!/_2 A\nu \operatorname{ctg}\vartheta\, (1 - \cos\vartheta) \tag{2.7}$$

which are obtained from the exact formulas at small values of the dimensionless parameter A connected with the heat source intensity by the following relation:

$$A = -\frac{\partial\sigma}{\partial T} \frac{Q}{2\pi\nu\eta\varkappa j}$$

We note that the expressions (2.7) deviate from the exact expressions by not more than 30% even at $A = 8$, and the larger values of A are, as we shall see below, of no interest.

3. Let us substitute (2.3), (2.4) and (2.7) into the system (2.6). After sufficiently lengthy manipulations we obtain the following system of two algebraic linear equations:

$$a_1 I_1 + a_2 I_3 = 0, \quad a_1 I_2 + a_2 I_4 = 0$$

with the condition of compatibility $I_1 I_4 = I_2 I_3$ determining the critical values of the dimensionless intensity A_* of the heat source. The integrals I_n $(n = 1 - 4)$ are

$$I_1 = A_* (-0.01709 m^4 + 0.2063 m^2 + 1.4615) + (0.02809 m^4 - 0.8280 m^2 + 3.400)$$

$$I_2 = A_* (-0.010496 m^4 + 0.1746 m^2 + 1.1183) + (0.01319 m^4 - 0.5464 m^2 + 2.6288)$$

$$I_3 = A_* (-0.010496 m^4 + 0.16116 m^2 + 1.6001) + (0.01319 m^4 - 1.3280 m^2 + 3.2568)$$

$$I_4 = A_* (-0.00695 m^4 + 0.14968 m^2 + 1.31215) + (0.00712 m^4 - 0.9679 m^2 + 3.056)$$

The spectrum of critical values $A_*^{(m)}$ for various values of $m = 1, 2, \ldots$ is

$$A_*^{(1)} = -3.4, \quad A_*^{(2)} = -1.5, \quad A_*^{(3)} = 0.18, \quad A_*^{(4)} = 2.0, \quad A_*^{(5)} = 3.8, \quad A_*^{(6)} = 5.2, \quad A_*^{(7)} = 6.4, \quad A_*^{(8)} = 7.0$$

etc. When $m \to \infty$, we have $A_*^{(m)} \to 7.72$.

4. The results obtained show that in the case of a "cold" source the axisymmetric motion can become unstable only relative to the vortical modes with $m = 1$ and $m = 2$. In the case of a "hot" source, the axisymmetric motion may become unstable relative to the vortical modes with $m \geqslant 3$. The critical values $A_*^{(m)}$ correspond to very low intensities (e.g. the source intensity

corresponding to $A = 7$ for water is only 10^{-4} watt). It follows that the axisymmetric flow will be disrupted by the vortical flow already in the presence of the low intensity sources. Since at $m > 7$ different values of m produce little difference in the values of $A_*^{(m)}$, it follows that the number of vortices is established in a practically random manner. A similar phenomenon was established experimentally in /2/.

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Translated by L.K.